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INTEGRAL EXPRESSIONS FOR TAIL PROBABILITIES OF THE MULTINOMIAL AND NEGATIVE MULTINOMIAL DISTRIBUTIONS*

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1. Summary and Introduction

Independent and identically distributed observations are taken, each of which assumes $k+2$ $(k \geq 1)$ possible values or attributes. We regard the observations as falling into one of $k+2$ possible mutually exclusive cells or categories $C_1,\ldots,C_{k+2}$ with respective probabilities $p_1,\ldots,p_{k+2}$, $\Sigma p_i = 1$. Let $s_1,\ldots,s_k$ denote non-negative integers, and let s denote a positive integer. Observations are taken one at a time until cell C_{k+1} contains s observations. It is assumed that $p_{k+1} > 0$ so that, with probability one, this sampling procedure terminates with a finite number of observations.

Denote the random number of observations in cell C_i at stopping time by X_j, $j = 1,\ldots,k$, and consider the following events:

E_1 : at the time of stopping, $X_j \geq s_j$ for all $j = 1,\ldots,k$,

E_2 : at the time of stopping, $X_j \leq s_j-1$ for all $j = 1,\ldots,k$.

If the s_j are all equal, then E_1 and E_2 refer to the minimum and maximum of $(X_1,\ldots,X_k)$, respectively. The probabilities $P\{E_1\}$ and

$P\{E_2\}$ are referred to as negative multinomial probabilities. In this paper we obtain integral expressions involving certain generalized F and Beta distributions for the straightforward summation formulas for $P\{E_1\}$, $P\{E_2\}$, and for a number of related probabilities. A number of recursion formulas and bounds are also developed.

Some of the multiple integrals also arise in the work of Gupta and Sobel (1962) dealing with the maximum and minimum of correlated F-statistics; in addition to exact computations, they develop approximations to these multiple integrals which depend on sums and products of (ordinary) incomplete Beta functions which have been tabulated. We mention that another class of generalized F and Beta distributions is discussed by Olkin (1959).

2. Integral Expressions

We first show that with no loss of generality the $(k+2)$-nd cell may be eliminated. Consider $P\{E_1\}$, namely,

$$P\{E_1\} = \sum_{\alpha=s}^{\infty} \sum_{x_1=s_1}^{\infty} \cdots \sum_{x_k=s_k}^{\infty} \frac{\Gamma(\alpha)\left(\prod_{i=1}^{k} p_i^{x_i}\right) p_{k+1}^{s}\, p_{k+2}^{\alpha-s-x_0}}{\Gamma(s)\,\Gamma(\alpha-s-x_0+1)\left(\prod_{1}^{k} x_i!\right)}, \tag{2.1}$$

where $x_0 = \sum_{1}^{k} x_i$. Interchange the order of the first summation symbol with the remaining k and let $j = \alpha-s-x_0$; then

$$P\{E_1\} = \sum_{x_1=s_1}^{\infty} \cdots \sum_{x_k=s_k}^{\infty} \frac{\Gamma(s+x_0)}{\Gamma(s)\prod_{1}^{k} x_i!} \left(\prod_{1}^{k} p_i^{x_i}\right) p_{k+1}^{s} \sum_{j=0}^{\infty} \frac{\Gamma(j+s+x_0)}{j!\,\Gamma(s+x_0)}\, p_{k+2}^{j}$$

(2.2)

$$= (1 - \sum_1^k \theta_i)^s \sum_{x_1=s_1}^{\infty} \cdots \sum_{x_k=s_k}^{\infty} \frac{\Gamma(s+x_0)}{\Gamma(s)\left(\prod_1^k x_i!\right)} \prod_1^k \theta_i^{x_i} ,$$

where $\theta_i = p_i/(1-p_{k+2})$, $i = 1,\ldots,k$. If we let $\theta_{k+1} = p_{k+1}/(1-p_{k+2})$, then $\sum_1^{k+1} \theta_i = 1$, and from (2.2) we see that the determination of $P\{E_1\}$ can be made from the $k+1$ probabilities $\theta_1,\ldots,\theta_{k+1}$.

Before obtaining the general result for (2.2), we consider the important special case $k=1$. (For convenience we omit the subscripts on x_1, θ_1, and s_1.) It is well known that for any real $s > 0$ and positive integer r,

$$(2.3) \quad (1-\theta)^s \sum_{x=r}^{\infty} \frac{\Gamma(s+x)}{\Gamma(s)\,x!} \theta^x = \frac{1}{B(r,s)} \int_0^{\theta} t^{r-1}(1-t)^{s-1}dt \equiv I_\theta(r,s)$$

$$= \frac{1}{B(r,s)} \int_0^{\frac{\theta}{1-\theta}} \frac{u^{r-1}}{(1+u)^{r+s}} du .$$

The LHS of (2.3) is generalized in (2.2), and we seek the corresponding generalization of the RHS of (2.3).

Remark. We note that, although in the present formulation s is necessarily an integer, in the final identity this need not be the case; for this reason the gamma function is used instead of the more natural factorial symbol.

2.1 Integral Identities for Upper Tail Probabilities

In this section we consider a generalization of (2.3) for the minimum problem in the case of the negative multinomial (Theorem 2.1) and the multinomial (Lemma 2.2) distributions.

For simplicity of notation let

$$B_m(a_1,\ldots,a_m) = \frac{\Gamma(a_1)\ \ldots\ \Gamma(a_m)}{\Gamma(a_1 + \ldots + a_m)} .$$

(We write $B(a_1,\ldots,a_m)$ when the "order" m is clear from the context.)

Theorem 2.1. If $s > 0$ is real, $k \geq 1$ and $s_1,\ldots,s_k$ are non-negative integers, $\theta_i \geq 0$, $i = 1,\ldots,k$, $\theta_{i+1} > 0$ and $\sum_1^{k+1} \theta_j = 1$, then

$$(2.4)\qquad \theta_{k+1}^s \sum_{x_1=s_1}^{\infty} \cdots \sum_{x_k=s_k}^{\infty} \frac{\Gamma(x_0+s)}{\Gamma(s)\prod_1^k x_i!} \prod_1^k \theta_i^{x_i}$$

$$= \frac{1}{B(s_1,\ldots,s_k,s)} \int_0^{\theta_1^*} \cdots \int_0^{\theta_k^*} \frac{\prod_1^k y_j^{s_j-1}}{(1 + \sum_1^k y_i)^{s+s_0}} \prod_1^k dy_j ,$$

where $x_0 = \sum_1^k x_i$, $s_0 = \sum_1^k s_i$, and $\theta_i^* = \theta_i/\theta_{k+1}$, $i = 1,\ldots,k$.

Remark. It is easily seen that the multiple sum on the left-hand side of (2.4) is convergent if and only if $\theta_{k+1} > 0$, i.e., $\sum_1^k \theta_i < 1$. When $\theta_{k+1} \to 0$ the RHS of (2.4) approaches unity, and in the limit we can use (2.4) as a definition of the LHS.

We also note that if $\sum_1^{k+1} \theta_j < 1$ then the result (2.4) still holds with $1 + \sum_1^k y_i$ in the integrand replaced by $c + \sum_1^k y_i$ where c is defined by $c\,\theta_{k+1} + \sum_1^k \theta_i = 1$; this remark also applies to theorem 2.4 below.

To prove (2.4) we first prove a lemma dealing with the "upper tail" of an ordinary multinomial distribution. In order to consider different cases at one time, let $m \geq 1$ denote the number of cells with $s_j > 0$, which we assume without loss of generality to be the first m cells.

Lemma 2.2: If $n, k, m,$ and $s_1,\ldots,s_m$ are positive integers with $1 \le m \le \min(k,n)$, $s_0 = \sum_1^m s_i \leqq n$, and if $p_1,\ldots,p_k$ are probabilities with $\sum_1^k p_i = 1$, then

$$(2.5)\quad \sum{}^{*} \frac{n!}{\prod_1^k x_i!} \prod_1^k p_i^{x_i} = \int_0^{p_1} \cdots \int_0^{p_m} \frac{\left[\prod_1^m t_i^{s_i-1}\right](1-t_0)^{n-s_0}}{B(s_1,\ldots,s_m,n-s_0+1)} \prod_1^m dt_i \ ,$$

where $t_0 = \sum_1^m t_i$, and the summation, Σ^*, is over all k-tuples $(x_1,\ldots,x_k)$ such that $\sum_1^m x_j = n$, $x_i \ge s_i$, $i = 1,\ldots,m$, and if $m < k$, $x_i \ge 0$, $i = m+1,\ldots,k$.

Proof of Lemma 2.2. We note that if $m=k$, then we need only write (2.5) for $k+1$ cells and let $p_{k+1} = 0$. For $m < k-1$ and $s_{m+1} = \cdots = s_k = 0$, it follows from

$$P\{X_i \ge s_i,\ i = 1,\ldots,k\} = P\{X_i \ge s_i,\ i = 1,\ldots,m\} \ ,$$

that we can make the correspondence $k'-1 = m$, $p'_{m+1} = \sum_{m+1}^k p_j$, $p'_i = p_i$, $i = 1,\ldots,m$, so that the problem is reduced to one with $m+1$ cells. Consequently, we need only consider the case $m = k-1$.

The proof is by induction on k. Let $Q(p_1)$ denote the left-hand side of (2.5) with $m = k-1$. Differentiating with respect to p_1 and simplifying by telescoping terms, we obtain

$$(2.6)\quad \frac{dQ}{dp_1} = \frac{p_1^{s_1-1}(1-p_1)^{n-s_1}}{B(s_1,n-s_1+1)} \sum{}^{*} \frac{(n-s_1)!}{\prod_{i=2}^k x_i!} \prod_{i=2}^k \left(\frac{p_i}{1-p_1}\right)^{x_i} \ ,$$

where the summation is over all $(k-1)$-tuples $(x_2,\ldots,x_k)$ with $\sum_2^k x_i = n-s_1$, $x_i \geq s_i$, $i = 2,\ldots,k$. Using the induction hypothesis for the $(k-1)$ fold expression in (2.6), we obtain

$$(2.7)\quad \frac{dQ}{dp_1} = p_1^{s_1-1}(1-p_1)^{n-s_1}\int_0^{q_2}\cdots\int_0^{q_{k-1}} \frac{\left[\prod_2^{k-1} u_i^{s_i-1}\right](1-\sum_2^{k-1} u_i)^{n-s_0-s_k}}{B(s_1,\ldots,s_{k-1},n-s_0+s_k+1)} \prod_1^{k-1} du_i,$$

where $s_0 = \sum_1^k s_j$, $q_i = p_i/(1-p_1)$, $i = 2,\ldots,k-1$. Transforming by $u_i = t_i(1-p_1)$ for $i = 2,\ldots,k-1$, and integrating with respect to p_1 results in

$$(2.8)\quad Q(p_1) = c + \int_0^{p_1}\cdots\int_0^{p_{k-1}} \frac{\left[\prod_1^{k-1} t_i^{s_i-1}\right](1-\sum_1^{k-1} t_i)^{n-s_0+s_k}}{B(s_1,\ldots,s_{k-1}, n-s_0+s_k+1)} \prod_1^{k-1} dt_i .$$

From $Q(0) = 0$ we have $c=0$. To complete the induction proof, we need only verify (2.5) for $k=2$, but this reduces to the well-known relation between the tail of the binomial distribution and the Incomplete Beta distribution. ||

<u>Proof of Theorem 2.1</u>: The left member of (2.4) may be written as

$$(2.9)\qquad P\{E_1\} = \sum_{\alpha=s_0}^{\infty} \frac{\Gamma(\alpha+s)}{\alpha!\,\Gamma(s)}\,\theta_0^{\alpha}(1-\theta_0)^s \sum{}^{*} \frac{\alpha!}{\prod_1^k x_i!}\prod_1^k \varphi_i^{x_i},$$

where $\theta_0 = \sum_1^k \theta_i$, $\varphi_i = \theta_i/\theta_0$, and the summation, Σ^*, is over all k-tuples $(x_1,\ldots,x_k)$ with $x_i \geq s_i$, $i = 1,\ldots,k$, and $\sum_1^k x_i = \alpha$.

Use of (2.5) in (2.9) and interchanging the order of summation and integration leads to

$$(2.10)\quad P\{E_1\} = \theta_0^{s_0}(1-\theta_0)^s \int_0^{\varphi_1}\cdots\int_0^{\varphi_k} \frac{\left[\prod_1^k t_i^{s_i-1}\right]}{B(s_1,\ldots,s_k,s)} \sum_{\alpha=s_0}^{\infty} \frac{\Gamma(s+\alpha)}{\Gamma(s+s_0)} \frac{[\theta_0(1-t_0)]^{\alpha-s_0}}{(\alpha-s_0)!} \prod_1^k dt_i,$$

where $t_0 = \sum_1^k t_j$. In (2.10) let $j = \alpha - s_0$ and sum the infinite series; then

$$(2.11)\quad P\{E_1\} = \frac{\theta_0^{s_0}(1-\theta_0)^s}{B(s_1,\ldots,s_k,s)} \int_0^{\varphi_1}\cdots\int_0^{\varphi_k} \frac{\prod_1^k t_i^{s_i-1}}{[1-\theta_0(1-t_0)]^{s+s_0}} \prod_1^k dt_i \;.$$

After simplifying and substituting $y_i = \theta_0 t_i/(1-\theta_0)$, $i = 1,\ldots,k$, the result (2.4) follows.

The case when any of the s_i are zero is proved by reducing both sides of (2.4) to the marginal probabilities which have the same structural form as in (2.4); the argument is given in detail in (2.22). ‖

2.2 Integral Identities for Lower Tail Probabilities

We can use Lemma 2.2 together with the Poincaré formula

$$(2.12)\quad P\{X_1 \le s_1-1,\ldots,X_k \le s_k-1\}$$
$$= 1 - \sum_{i=1}^{k} P\{X_i \ge s_i\} + \sum_{i,j=1}^{k} P\{X_i \ge s_i,\ X_j \ge s_j\} - \cdots$$

to find integral expressions for lower tail probabilities. If the sum of the s_i inside the braces of any term on the right-hand side of

(2.12) is greater than n, then these terms are zero and can be removed at the outset. The remaining terms can then be written as integrals by using Lemma 2.2.

If $s_0 = s_1 + \cdots + s_k \leq n$, then none of the terms are deleted, and we have as a direct consequence of (2.5) that

$$(2.13) \quad P\{X_1 \leq s_1-1, \ldots, X_k \leq s_k-1\}$$

$$= 1 - \sum_{i=1}^{k} P\{T_i < p_i\} + \sum_{i,j=1}^{k} P\{T_i < p_i,\ T_j < p_j\} - \cdots ,$$

where the random variables $T_1, \ldots, T_k$ have the joint density

$$(2.14) \qquad f(t_1, \ldots, t_k) = \frac{\left[\prod_1^k t_i^{s_i-1}\right] (1 - \sum_1^k t_i)^{n-s_0}}{B(s_1, \ldots, s_k,\ n-s_0+1)} ,$$

for $0 < t_i$, $i = 1, \ldots, k$; $\sum_1^k t_i < 1$.

Thus, by applying Lemma 2.2 to each term of (2.13), we obtain the following result for the lower tail of the multinomial distribution.

If n, k, and $s_1, \ldots, s_k$ are positive integers with $s_0 = s_1 + \cdots + s_k \leq n$, and $0 \leq p_j \leq 1$, $j = 1, \ldots, k$, $\sum_1^k p_j = 1$, then

$$(2.15) \quad \Sigma^* \frac{n!}{\prod_1^k x_i!} \prod_1^k p_i^{x_i}$$

$$= \int_{p_1}^{1 - \sum_2^k p_j} \int_{p_2}^{1-t_1-\sum_3^k p_j} \cdots \int_{p_k}^{1 - \sum_1^{k-1} t_j} \frac{\left[\prod_1^k t_i^{s_i-1}\right] (1-t_0)^{n-s_0}}{B(s_1, \ldots, s_k, n-s_0+1)} \prod_1^k dt_i ,$$

where $t_0 = \sum_1^k t_i$, and Σ^* denotes summation over all k-tuples

$(x_1,\dots,x_k)$ with $0 \leq x_i \leq s_i-1$, $i = 1,\dots,k$, and $\sum_1^k x_j = n$.

If $k-m$ of the s_j are equal to or greater than $n+1$, and the remaining m of the s_j (say, $s_1,\dots,s_m$) have a sum $s_1 + \cdots + s_m = n$, then the terms containing an $s_j = n+1$ are deleted from (2.12), and (2.13) takes the form

$$(2.16)\quad P\{X_1 \leq s_1-1,\dots,X_m \leq s_m-1\}$$

$$= 1 - \sum_{i=1}^m P\{T_i < p_i\} + \sum_{i,j=1}^m P\{T_i < p_i,\ T_j < p_j\} - \cdots,$$

where if $m < k$, the joint density of $T_1,\dots,T_m$ is obtained from (2.14) by integrating over $0 < T_j$, $j = m+1,\dots,k$, $\sum_{m+1}^k T_j < 1 - \sum_1^m T_j$, in which case (2.15) may be stated in a more general context.

<u>Lemma 2.3</u>: If m, k, n, and $s_1,\dots,s_k$ are positive integers with $1 \leqq m \leqq \min(k,n)$, and $s_0 = s_1 + \cdots + s_m \leq n$, then

$$(2.17)\quad \Sigma^* \frac{n!}{\prod_1^k x_i!} \prod_1^k p_i^{x_i}$$

$$= \int_{p_1}^{1-\sum_2^m p_i} \int_{p_2}^{1-t_1-\sum_3^m p_i} \cdots \int_{p_m}^{1-\sum_1^{m-1} t_i} \frac{\left[\prod_1^m t_i^{s_i-1}\right](1-t_0)^{n-s_0}}{B(s_1,\dots,s_m,\ n-s_0+1)} \prod_1^m dt_i\ ,$$

where $t_0 = \sum_1^m t_i$, and Σ^* denotes the summation over all k-tuples with $0 \leq x_j \leq s_j-1$, $j = 1,\dots,m$, and $\sum_1^m x_j = n$. It is interesting to note that the RHS of (2.17) depends only on $(p_1,\dots,p_m)$ and $(s_1,\dots,s_m)$, and not on $(p_{m+1},\dots,p_k)$ or $(s_{m+1},\dots,s_k)$.

From lemmas 2.2 and 2.3 we can now obtain integral expressions for the distribution of the minimum and the maximum of $X_1, X_2, \ldots, X_m$ by setting $s_1 = s_2 = \ldots = s_m = s$ (say) where s is a positive integer. For the particular value $n = s_o \equiv ms$ we obtain from (2.5) the interesting simple result for the minimum

$$P\{\min_{1 \le i \le m} X_i \ge s\} = \frac{(ms)!}{(s!)^m} \left(\prod_{1}^{m} p_i\right)^s ;$$

there is no analogous result for the maximum.

To obtain a result for lower tail probabilities which is dual to Theorem 2.1, we require the marginal probabilities of the expression in (2.4) when $k \geq 2$. Define

$$L(s_1,\ldots,s_k;\ s|\theta_1,\ldots,;\ \theta_{k+1}) \equiv \theta_{k+1}^{s} \sum_{x_1=s_1}^{\infty} \cdots \sum_{x_k=s_k}^{\infty} \frac{\Gamma(x_0+s)}{\Gamma(s)\prod_1^k x_j!} \prod_1^k \theta_i^{x_i} ,$$

where $x_0 = \sum_1^k x_i$. Using the identity

$$(1-\theta)^{-a} = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{\Gamma(a)} \quad \frac{\theta^j}{j!} ,$$

it follows by a direct calculation that

$$\begin{aligned} (2.18) \qquad & L(0,s_2,\ldots,s_k;\ s|\theta_1,\ldots,\theta_k;\ \theta_{k+1}) \\ & = L(s_2,\ldots,s_k;\ s|\ \frac{\theta_2}{1-\theta_1},\ldots,\frac{\theta_k}{1-\theta_1};\ \frac{\theta_{k+1}}{1-\theta_1}) . \end{aligned}$$

By an iteration of (2.18), we obtain

$$\begin{aligned} (2.19) \qquad & L(0,\ldots,0,s_m,\ldots,s_k;\ s|\theta_1,\ldots,\theta_k;\ \theta_{k+1}) \\ & = L(s_m,\ldots,s_k;\ s|\ \frac{\theta_m}{1-\sum_1^{m-1}\theta_j},\ldots,\frac{\theta_k}{1-\sum_1^{m-1}\theta_j};\ \frac{\theta_{k+1}}{1-\sum_1^{m-1}\theta_j}) , \end{aligned}$$

so that the marginal probabilities are of the same structural form as the LHS of (2.4).

Define the RHS of (2.4) by

$$R(s_1,\ldots,s_k;\ s|\rho_1,\ldots,\rho_k) \equiv \frac{1}{B(s_1,\ldots,s_k,s)} \int_0^{\rho_1}\cdots\int_0^{\rho_k} \frac{\prod_1^k y_j^{s_j-1}}{(1+\sum_1^k y_i)^{s+s_0}} \prod_1^k dy_i .$$

Integration over y_1 yields

$$(2.20) \qquad R(\infty, s_2,\ldots,s_k;\ s|\rho_1,\ldots,\rho_k) = R(s_2,\ldots,s_k;\ s|\rho_2,\ldots,\rho_k);$$

by iterating (2.20) we obtain

$$(2.21) \quad R(\infty,\ldots,\infty,\ s_m,\ \ldots\ ,\ s_k;\ s|\rho_1,\ldots,\rho_k) = R(s_m,\ldots,s_k;\ s|\rho_m,\ldots,\rho_k),$$

and again the marginal probabilities are at the same structural form as the RHS of (2.4).

For a typical marginal probability, using (2.19), (2.4), and (2.21), it follows that

$$(2.22) \quad L(0,\ldots,0,s_m,\ldots,s_k;\ s|\theta_1,\ldots,\theta_k;\ \theta_{k+1})$$

$$= L\left(s_m,\ldots,s_k;\ s\left|\frac{\theta_m}{1-\sum_1^{m-1}\theta_i},\ldots,\frac{\theta_k}{1-\sum_1^{m-1}\theta_i};\ \frac{\theta_{k+1}}{1-\sum_1^{m-1}\theta_i}\right.\right)$$

$$= R\left(s_m,\ldots,s_k;\ s\ \left|\ \frac{\theta_m}{\theta_{k+1}},\ldots,\frac{\theta_k}{\theta_{k+1}}\right.\right)$$

$$= R\left(\infty,\ldots,\infty,\ s_m,\ldots,s_k;\ s\left|\frac{\theta_1}{\theta_{k+1}},\ldots,\frac{\theta_k}{\theta_{k+1}}\right.\right)$$

$$= R(\infty,\ldots,\infty,\ s_m,\ldots,s_k;\ s|\theta_1^*,\ \ldots\ ,\ \theta_k^*)\ .$$

Thus, with θ_i^* defined independently of m as in Theorem 2.1, the operation of setting certain $s_i = 0$ in the LHS of (2.4) is equivalent to setting corresponding $\theta_i^* = \infty$ in the RHS of (2.4). We use the result in the proof of the following theorem on the lower tail of the negative multinomial distribution.

<u>Theorem 2.4</u>. If $s > 0$ is real, k and $s_1, \ldots, s_k$ are positive integers, $\theta_i \geq 0$, $i = 1, \ldots, k$, $\theta_{k+1} > 0$, $\sum \theta_i = 1$, then

$$(2.23) \qquad \theta_{k+1}^{s} \sum_{x_1=0}^{s_1-1} \cdots \sum_{x_k=0}^{s_k-1} \frac{\Gamma(s+x_0)}{\Gamma(s) \prod_1^k x_i!} \prod_1^k \theta_i^{x_i}$$

$$= \frac{1}{B(s_1, \ldots, s_k, s)} \int_{\theta_1^*}^{\infty} \cdots \int_{\theta_k^*}^{\infty} \frac{\prod_1^k y_i^{s_i-1}}{(1+\sum_1^k y_i)^{s+s_0}} \prod dy_i \,,$$

where $x_0 = \sum_1^k x_i$, $s_0 = \sum_1^k s_i$, $\theta_i^* = \theta_i/\theta_{k+1}$, $i = 1, \ldots, k$.

<u>Proof</u>: Let X_i denote the number of observations in the i-th cell and let A_i denote the event $X_i \geq s_i$ at the time of stopping. Let B_i denote the event that $Y_i < \theta_i^*$, where $Y_1, \ldots, Y_k$ have a joint density given by (2.14). From (2.22), $P\{A_{i_1} \cdots A_{i_r}\} = P\{B_{i_1} \cdots B_{i_r}\}$ for any r, and hence

$$\begin{aligned}
&P\{\text{at least one } X_i \geq s_i\} = P\{\cup A_i\} \\
&= \sum_1^k P\{A_i\} - \sum_{i<j}^{k} P\{A_i A_j\} + \cdots + (-1)^{k-1} P\{A_1 \cdots A_k\} \\
&= \sum_1^k P\{B_i\} - \sum_{i<j}^{k} P\{B_i B_j\} + \cdots + (-1)^{k-1} P\{B_1 \cdots B_k\} \\
&= P\{\cup B_i\} = P\{\text{at least one } Y_i < \theta_i^*\} \,.
\end{aligned}$$

The result follows by taking complements. $\|$

Remark: Although the main purpose in writing (2.23) is that for $k \geq 2$ and large values of s_i the integral in (2.23) can be used to evaluate the sum on the left, for small values of the s_i, we may use the sum to evaluate the integral. For the latter purpose, note that $\theta_j^* = \theta_j/\theta_{k+1}$, $j = 1,\ldots, k$, $\sum_1^{k+1} \theta_j = 1$, from which we calculate $\theta_j = \theta_j^*/(1 + \sum_1^k \theta_j^*)$, $j = 1,\ldots,k$ and $\theta_{k+1} = (1 + \sum_1^k \theta_j^*)^{-1}$, which are now used for the LHS of (2.23).

In this connection the following bound may also be useful. Let $s_1 = \min(s_1,\ldots,s_k)$ and let P_1 denote the sum of all terms in the left member of (2.23) for which $x \equiv x_1 + \cdots + x_k \leq s_1-1$. Then

$$(2.24)\quad P_1 = \sum_{x=0}^{s_1-1} \frac{\Gamma(s + x)}{\Gamma(s)\ x!}\, \theta_{k+1}^{s}(1-\theta_{k+1})^x \sum \frac{x!}{\prod_1^k x_i!} \prod_1^k \left(\frac{\theta_j}{1-\theta_{k+1}} \right),$$

where the inside summation is a complete multinomial sum and hence equals unity. From (2.3) we have

$$(2.25)\qquad P_1 = I_{\theta_{k+1}}(s, s_1) ,$$

where $\theta_{k+1} = (1 + \sum_1^k \theta_j^*)^{-1}$. Clearly, (2.25) is a lower bound for $P\{E_1\}$ which can also be used as an exact expression for part of the summation of $P\{E_1\}$.

3. Recursion Formulas.

We now develop some recursion formulas which provide an alternative method for the computation of the various probabilities. Denote the

LHS of (2.17) by $J_m\begin{pmatrix} s_1,\ldots,s_m;\ n \\ p_1,\ldots,p_m;\ k\end{pmatrix}$; the subscript on J is omitted whenever the "order" m is clear from the context. In terms of the urn model, this refers to the probability that for m of the cells (say, the first m cells) $x_i \le s_i - 1$, $1 \le s_i \le n$, $i = 1,\ldots, m$. For the remaining $k-m$ cells, we can integrate out the x_i corresponding to $k-m-1$ of these cells leaving one cell (say, the $(m+1)$-st cell) with probability $p_o = 1 - \sum_1^m p_i$; thus for $k \ge m+1$

$$(3.1)\qquad J\begin{pmatrix} s_1,\ldots,s_m;\ n \\ p_1,\ldots,p_m;\ k\end{pmatrix} = J\begin{pmatrix} s_1,\ldots,s_m;\ n \\ p_1,\ldots,p_m;\ m+1\end{pmatrix}.$$

There are now various ways to generate recursion formulas. Integration by parts in (2.17) results in two reduction formulas

$$(3.2)\qquad J\begin{pmatrix} s_1,\ldots,s_m;\ n \\ p_1,\ldots,p_m;\ m+1\end{pmatrix}$$

$$= \binom{n}{s_m-1} p_m^{s_m-1}(1-p_m)^{n-s_m+1}\, J\begin{pmatrix} s_1,\ldots,s_{m-1};\ n-s_m+1 \\ \theta_1,\ldots,\theta_{m-1};\ m\end{pmatrix}$$

$$+\, J\begin{pmatrix} s_1,\ldots,s_{m-1},\ s_m-1;\ n+2 \\ p_1,\ldots,p_{m-1},\ p_m;\ m+1\end{pmatrix}, \text{ for } s_m \ge 1$$

$$(3.2)\qquad J\begin{pmatrix} s_1,\ldots,s_m;\ n \\ p_1,\ldots,p_m;\ m+1\end{pmatrix}$$

$$= -\binom{n}{s_m} p_m^{s_m}(1-p_m)^{n-s_m}\, J\begin{pmatrix} s_1,\ldots,s_{m-1};\ n-s_m \\ \theta_1,\ldots,\theta_{m-1};\ m\end{pmatrix}$$

$$+\, J\begin{pmatrix} s_1,\ldots,s_{m-1},\ s_m+1;\ n \\ p_1,\ldots,p_{m-1},\ p_m;\ m+1\end{pmatrix} \text{ for } n \ge \sum_1^m s_i\,,$$

where the $\theta_j \equiv p_j/(1-p_m)$, $j = 1,\ldots, m-1$, sum to unity. In the above definition, the J function is zero if any $s_j = 0$. Formulas (3.2) and (3.3) are especially useful when one or more of the s_i are close to unity, and n, respectively, in which case the second term on the right hand side of (3.2) and of (3.3) vanishes. This procedure permits a reduction from an m-dimensional integral to a sum of a small number of integrals, each of which is at most (m-1)-dimensional. Upon iteration, we eventually obtain a sum of tabulated incomplete Beta function. This method is exact, and is useful if the dimensionality of the basic integral is not too large.

If we expand $J\begin{pmatrix} s_1,\ldots,s_m; & n \\ p_1,\ldots,p_m; & m+1 \end{pmatrix}$ with respect to the number of entries in the (m+1)-st cell, we obtain

$$(3.4)\qquad J\begin{pmatrix} s_1,\ldots,s_m; & n \\ p_1,\ldots,p_m; & m+1 \end{pmatrix} = \sum_{\alpha=0}^{s_o-m} \binom{n}{\alpha} (1-p_o)^{\alpha}\, p_o^{n-\alpha}\, J\begin{pmatrix} s_1,\ldots,s_m; & \alpha \\ p_1^*,\ldots,p_m^*; & m \end{pmatrix},$$

where $s_o = \sum_1^m s_j$, $p_j^* = p_j/(1-p_o)$, $j = 1, \ldots, m$.

Similarly, an expansion with respect to the number of entries in the m-th cell yields

$$(3.5)\qquad J\begin{pmatrix} s_1,\ldots,s_m; & n \\ p_1,\ldots,p_m; & m+1 \end{pmatrix} = \sum_{\beta=n-s_m+1}^{n} \binom{n}{\beta} (1-p_m)^{\beta}\, p_m^{n-\beta}\, J\begin{pmatrix} s_1,\ldots,s_{m-1}; & \beta \\ \hat{p}_1,\ldots,\hat{p}_{m-1}; & m \end{pmatrix}$$

where $\hat{p}_j = p_j/(1-p_m)$, $j = 1, \ldots, m-1$.

Any of these recursion formulas can be iterated until $m = 1$, in which case

$$J\begin{pmatrix} a; & n \\ p; & 2 \end{pmatrix} = I_{1-p}(n-a+1,\, a)\ ,$$

so that the basic integrals have a representation as a linear combination of incomplete Beta functions.

In either (3.4) or (3.5) we note that the RHS is of the form

$$(3.6)\qquad \sum_{\alpha=0}^{A} a_\alpha J\begin{pmatrix} s_1,\dots,s_m; & \alpha \\ p_1,\dots,p_m; & m+1 \end{pmatrix} \equiv \sum_{\alpha=0}^{A} a_\alpha\, g(\alpha)\ ,$$

where $a_\alpha \geqq 0$ and $\Sigma a_\alpha \leqq 1$. Consequently, we immediately obtain bounds in terms of $\max_\alpha g(\alpha)$ or $\min_\alpha g(\alpha)$. However, it is clear that $g(\alpha)$ is a monotone non-increasing function on the positive integers, for if the inequalities $X_j \leqq s_j - 1$, $j = 1,\dots,m$, holds for the first $\alpha + 1$ observations, then they hold a fortiori for the first α observations.

More generally, we can iterate the expansion in (3.5) and use the monotonicity to prove that

$$(3.7)\qquad J\begin{pmatrix} s_1,\dots,s_{m-r} & ; & n \\ p_{1r},\dots,p_{m-r,r} & ; & m-r+1 \end{pmatrix} J\begin{pmatrix} s_{m-r+1},\dots,s_m; & n \\ p_{m-r+1},\dots,p_m; & r+1 \end{pmatrix}$$

$$\leqq J\begin{pmatrix} s_1,\dots,s_m; & n \\ p_1,\dots,p_m; & m+1 \end{pmatrix}$$

$$\leqq J\begin{pmatrix} s_1,\dots,s_{m-r} & ; n+r - \sum_{j=0}^{r-1} s_{m-j} \\ p_{1r},\dots,p_{m-r,r}; & m-r+1 \end{pmatrix} \quad J\begin{pmatrix} s_{m-r+1},\dots,s_m; & n \\ p_{m-r+1},\dots,p_m; & r+1 \end{pmatrix}$$

where $p_{ir} = p_i/(1 - \sum_{j=0}^{r-1} p_{m-j})$.

Thus we find from (3.7) for $r = 1$, that

$$(3.8) \qquad I_{1-p_m}(n - s_m+1,\ s_m)\ J\begin{pmatrix} s_1,\ldots,s_{m-1} & ;\ n \\ p_{11},\ldots,p_{n-1,1}; & m \end{pmatrix}$$

$$\leq J\begin{pmatrix} s_1,\ldots,s_m; & n \\ p_1,\ldots,p_m; & m+1 \end{pmatrix}$$

$$\leq I_{1-p_m}(n-s_m+1,\ s_m)\ J\begin{pmatrix} s_1,\ldots,s_{m-1} & ;\ n-s_m+1 \\ p_{11},\ldots,p_{m-1,1}; & m \end{pmatrix},$$

from which by iteration, we obtain

$$(3.9) \qquad I_{1-p_m}(n-s_m+1,s_m)\ I_{\frac{1-p_{m-1}-p_m}{1-p_m}}(n-s_{m-1}+1,\ s_{m-1}) \cdots I_{\frac{1-p_1-\cdots-p_m}{1-p_2-\cdots-p_m}}(n-s_1+1,s_1)$$

$$\leq J\begin{pmatrix} s_1,\ldots,s_m; & n \\ p_1,\ldots,p_m; & m+1 \end{pmatrix}$$

$$\leq I_{1-p_m}(n-s_m+1,s_m)\ I_{\frac{1-p_{m-1}-p_m}{1-p_m}}(n-s_{m-1}-s_m+2,s_{m-1})$$

$$\cdots\ I_{\frac{1-p_1-\cdots-p_m}{1-p_2-\cdots-p_m}}(n-s_1-\cdots-s_m+m,\ s_1)\ .$$

In a similar manner, letting $I\begin{pmatrix} s_1,\ldots,s_m; & n \\ p_1,\ldots,p_m; & m+1 \end{pmatrix}$ denote the left member of (2.5), and integrating by parts we obtain the reduction formulas

$$(3.10) \quad I\begin{pmatrix} s_1,\ldots,s_m; & n \\ p_1,\ldots,p_m; & m+1 \end{pmatrix} = -\binom{n}{s_m-1}\ p_m^{\,s_m-1}\ (1-p_m)^{n-s_m+1}\ I\begin{pmatrix} s_1,\ldots,s_{m-1}; & n-s_m+1 \\ \theta_1,\ldots,\theta_{m-1}; & m \end{pmatrix}$$

$$+\ I\begin{pmatrix} s_1,\ldots,s_{m-1},s_m-1; & n \\ p_1,\ldots,p_{m-1},p_m & ;\ m+1 \end{pmatrix}$$

$$(3.11)\qquad I\begin{pmatrix} s_1,\dots,s_m; & n \\ p_1,\dots,p_m; & m+1 \end{pmatrix} = \binom{n}{s_m} p_m^{s_m} (1-p_m)^{n-s_m} I\begin{pmatrix} s_1,\dots,s_{m-1}; & n-s_m \\ \theta_1,\dots,\theta_{m-1}; & m \end{pmatrix}$$

$$+ \; I\begin{pmatrix} s_1,\dots,s_{m-1},s_m+1; & n \\ p_1,\dots,p_{m-1},p_m\,; & m+1 \end{pmatrix}$$

where θ_i is as defined above. Clearly,

$$(3.12)\qquad I\begin{pmatrix} s;n \\ p;2 \end{pmatrix} = I_p(s,n-s+1) = 1 - J\binom{s,n}{p,2} .$$

Corresponding to (3.4) and (3.5), we now obtain, in terms of p_0, p_j^* and $\hat{p}_j$ $(j = 1,\dots,m)$ all defined above,

$$(3.13)\qquad I\begin{pmatrix} s_1,\dots,s_m; & n \\ p_1,\dots,p_m; & m+1 \end{pmatrix} = \sum_{\alpha=s_0}^{n} \binom{n}{\alpha} (1-p_0)^{\alpha} p_0^{n-\alpha} I\begin{pmatrix} s_1,\dots,s_m; & \alpha \\ p_1^*,\dots,p_m^*; & m \end{pmatrix}$$

$$(3.14)\qquad I\begin{pmatrix} s_1,\dots,s_m; & n \\ p_1,\dots,p_m; & m+1 \end{pmatrix} = \sum_{\beta=S_{m-1}}^{n-s_m} \binom{n}{\beta} (1-p_m)^{\beta} p_m^{n-\beta} I\begin{pmatrix} s_1,\dots,s_{m-1}; & \beta \\ \hat{p}_1,\dots,\hat{p}_{m-1}; & m \end{pmatrix}$$

where $S_{m-1} = \sum_{i=1}^{m-1} s_i$.

Using the fact that the left member of (2.5) is non-decreasing in n we obtain corresponding to the bounds in (3.7)

$$(3.15)\quad I\begin{pmatrix} s_1,\dots,s_{m-r}; & s_1+ \dots +s_{m-r} \\ p_{1r},\dots,p_{m-r,r}; & m-r+1 \end{pmatrix} \quad I\begin{pmatrix} s_{m-r+1},\dots,s_m; & n \\ p_{m-r+1},\dots,p_m; & r+1 \end{pmatrix}$$

$$\leq I\begin{pmatrix} s_1,\dots,s_m; & n \\ p_1,\dots,p_m; & m+1 \end{pmatrix}$$

$$\leq I\begin{pmatrix} s_1,\dots,s_{m-r}; & n-s_{m-r+1}-\dots-s_m \\ p_{1r},\dots,p_{m-r,r}; & m-r+1 \end{pmatrix} \; I\begin{pmatrix} s_{m-r+1},\dots,s_m; & n \\ p_{m-r+1},\dots,p_m; & r+1 \end{pmatrix}$$

where p_{ir} is defined above. For r = 1 this gives

$$(3.16)\qquad I_{p_m}(s_m, n-s_m+1)\; I\begin{pmatrix} s_1,\ldots,s_{m-1};\ s_1+\ldots+s_{m-1} \\ p_{11},\ldots,p_{m-1,1};\ m \end{pmatrix}$$

$$\leq I\begin{pmatrix} s_1,\ldots,s_m;\ n \\ p_1,\ldots,p_m;\ m+1 \end{pmatrix}$$

$$\leq I_{p_m}(s_m, n-s_m+1)\quad I\begin{pmatrix} s_1,\ldots,s_{m-1};\ n-s_m \\ p_{11},\ldots,p_{m-1,1};\ m \end{pmatrix},$$

from which by iteration, we obtain

$$(3.17)\quad I_{p_m}(s_m, n+1-s_m)\; I_{\frac{p_{m-1}}{1-p_m}}\left(s_{m-1}, 1+\sum_1^{m-2} s_i\right) \ldots I_{\frac{p_1}{1-p_2-\ldots-p_m}}(s_1, 1)$$

$$\leq I\begin{pmatrix} s_1,\ldots,s_m;\ n \\ p_1,\ldots,p_m;\ m+1 \end{pmatrix}$$

$$\leq I_{p_m}(s_m, n+1-s_m)\; I_{\frac{p_{m-1}}{1-p_m}}(s_{m-1}, n+1-s_m-s_{m-1}) \ldots I_{\frac{p_1}{1-p_2-\ldots-p_m}}\left(s_1, n+1-\sum_1^m s_i\right).$$

4. Other Tail Probabilities.

In the basic urn model we continue taking observations until cell C_{k+1} contains s observations. Our concern then was with the events that each cell C_j contains at least s_j or at most s_j-1 observations. There are, of course, other models and events which may be of interest.

We now consider a different event with the same model. Suppose the k cells are ordered and the event of interest is that at the time of stopping there are at least s_1, $s_1 + s_2$, ... , $s_1 + \dots + s_k$ observations in cells C_1, the union of C_1 and C_2, ..., the union of C_1 and C_2 and ... and C_k, respectively. This is a special type of cumulative model, but others may be handled in a similar manner.

Corresponding to the tail probabilities of Section 2, we now have

$$c(s_1,\dots,s_k;p_1,\dots,p_k;\ n) = \Sigma \frac{n!}{\prod_1^k x_j!} \prod p_j^{x_j} ,$$

where the summation is over all k-tuples $(x_1,\dots,x_k)$ with

$$(4.1) \qquad x_1 \geqq s_1, x_1 + x_2 \geqq s_1 + s_2, \dots, \sum_1^{k-1} x_j \geqq \sum_1^{k-1} s_j, \quad \sum_1^{k-1} s_i \leqq n .$$

By an argument similar to that in the proof of Lemma 2.2, we show that

$$(4.2) \qquad c(s_1,\dots,s_k;\ p_1,\dots,p_k;\ n)$$

$$= [B(s_1,\dots,s_k;\ n-s_o+1)]^{-1}$$

$$\times \int_o^{\delta_1} \int_o^{\delta_2 - v_1} \dots \int_o^{\delta_{k-1} - \sum_1^{k-2} v_i} \left[\prod_1^{k-1} v_j^{s_j - 1} \right] (1-v_o)^{n-s_o} \prod_1^{k-1} dv_j ,$$

where $\delta_j = p_1 + \dots + p_j$, $j = 1, \dots, k-1$, $v_o = \sum_1^{k-1} v_i$, $s_o = \sum_1^{k-1} s_i$.

To prove (4.2) consider $c(s_1,\dots,s_k;p_1,\dots,p_k;\ n) = Q(p_k)$ as a function of p_k with $p_1,\dots,p_{k-2}$ free, $p_{k-1} = 1 - \sum_1^{k-2} p_i - p_k$.

After differentiating $Q(p_k)$, collapsing sums, and simplifying, we obtain

$$(4.3)\qquad \frac{dQ(p_k)}{dp_k} = \frac{-(1-p_k)^{\sum_1^{k-1} s_i - 1}}{B(S_{k-1}, n+1-S_{k-1})}\, c\left(s_1,\ldots,s_{k-1};\ \frac{p_1}{1-p_k},\ldots,\frac{p_{k-1}}{1-p_k};\ \sum_1^{k-1} s_i - 1\right).$$

where $S_{k-1} = s_1 + \ldots + s_{k-1}$. Using induction for k-1, (4.3) can be expressed as a (k-2)-fold integral. Integration with respect to p_k (using the fact that $Q(1) = 0$) leads to (4.2). The proof is completed by noting that for $k = 2$, $c(s_1,s_2;p_1,p_2;n)$ reduces to (2.5), which completes the induction argument.

Remark. There are various ways to prove (4.2). The following method is an alternative which has intrinsic interest, and we sketch the underlying idea. Consider n independent observations on a uniform distribution on [0,1]. The points $\delta_1 = p_1$, $\delta_2 = p_1 + p_2$, ..., $\delta_{k-1} = p_1 + \ldots + p_{k-1}$, divide the unit interval into k subintervals (cells) of length $p_1,\ldots,p_k$. If X_j denotes the number of observations that falls in the j-th cell, then the probability that $X_1 \geqq s_1$, $X_1 + X_2 \geqq s_1 + s_2,\ldots,X_1 + \ldots + X_k \geqq s_1 + \ldots + s_k$ is given by the LHS of (4.2). On the other hand, if we let T_j denote the length of the interval from the left endpoint of the j-th cell to the s_j-th order statistic in the j-th cell, then the random variables $T_1, \ldots, T_{k-1}$ have the multivariate Beta distribution as given by (2.14). The condition $\sum_1^r X_j \geqq \sum_1^r s_j$, $r = 1,\ldots,k-1$, is now equivalent to the condition

$\sum_1^r T_j < \sum_1^r p_j$, $r = 1,\ldots,k-1$. Integrating (2.14) over $\sum_1^r t_j < \sum_1^r p_j$, $r = 1,\ldots,k-1$, then yields (4.1).

Now suppose we have k+1 cells with probabilities $p_1,\ldots,p_{k+1}$ summing to unity and with $p_{k+1} > 0$; let $0 \le b_1 \le \cdots \le b_k < \infty$ be non-negative integers. Observations are taken one at a time until $X_{k+1} = s$. Consider the probability that at a time of stopping, $X_1 + \cdots + X_j \ge b_j$, $j = 1,\ldots, k$. Letting $\delta_j^* = \delta_j/p_{k+1}$ $= (p_1 + \cdots + p_j)/p_{k+1}$, we obtain

<u>Theorem 4.1.</u>

$$(4.3)\quad \sum \frac{(s + \sum_1^k x_j - 1)!}{(s-1)! \prod_1^k x_i!} \prod_1^k p_i^{x_i}\, p_{k+1}^s$$

$$= \frac{(s+b_k-1)!}{(s-1)! \prod_1^k (b_i-b_{i-1})!} \int_0^{\delta_1^*} \int_0^{\delta_2^*-v_1} \cdots \int_0^{\delta_{k-1}^* - \sum_1^{k-2} v_i} \int_0^{p_k(\delta_k^* - \sum_1^{k-1} v_i)}$$

$$\times \frac{\left[\prod_1^k v_i^{b_i-b_{i-1}-1}\right]}{(1+\sum_1^{k-1} v_i)^{s+b_{k-1}}} (1-v_k)^{s+b_{k-1}-1} \prod_1^k dv_i,$$

where $b_o = 0$.

Proof: Rewrite the LHS of (4.4) as

$$\sum_{y_k=b_k}^{\infty} \frac{(s+y_k-1)!}{(s-1)!\, y_k!} \delta_k^{y_k} (1-\delta_k)^s \sum \frac{y_k!}{\prod_1^k x_i!} \prod_1^k \left(\frac{p_k}{\delta_k}\right)^{x_i},$$

where the inside sum is over all k-tuples $(x_1,\ldots,x_k)$ for which $\sum_1^k x_j = y_k$, $\sum_1^r x_i \geq b_r$, $r = 1,\ldots,k-1$. Hence by Lemma 2.2, the above sum is equal to

$$\sum_{z=b_k-b_{k-1}}^{\infty} \frac{(s+b_{k-1}-1)!}{(s-1)!\prod_1^{k-1}(b_j-b_{j-1}-1)!} \frac{(s+b_{k-1}+z-1)!}{(s+b_{k-1}-1)!\, z!} \delta_k^{b_{k-1}} (1-\delta_k)^s$$

$$\times \int_0^{\frac{\delta_1}{\delta_k}} \int_0^{\frac{\delta_2}{\delta_k}-u_1} \cdots \int^{\frac{\delta_{k-1}}{\delta_k}-\sum_1^{k-2} u_i} \left[\prod_1^{k-1} u_i^{b_i-b_{i-1}-1}\right]\left[\delta_i(1-\sum_1^{k-1} u_i)\right]^z \prod_1^{k-1} du_i \,.$$

Interchanging the order of integration and summation, and using (2.3) to sum the series, and letting $u_j = [\delta_k/(1-\delta_k)]\, v_j$, $j = 1,\ldots,k-1$, we obtain (4.4).||

The result (4.2) for the special parallelopiped type regions of summation (4.1) can be slightly generalized. Suppose for any one value of i, say i_o, we replace the restriction $x_{i_o} \geq s_{i_o}$ in (2.5) by a new restriction (say) $x_{i_o} + x_\alpha + x_\beta \geq s_{i_o} + s_\alpha + s_\beta$ without altering any of the other restrictions, $x_j \geq s_j$ $(j \neq i_o)$. Assuming that $i_o < \alpha < \beta$ (for convenience), we can then assert that the resulting sum is equal to a multiple integral of the same density as in the right member of (2.5) except that the upper limit of integration for t_i is taken from the inequality $t_{i_o} + t_\alpha + t_\beta < p_{i_o}+p_\alpha+p_\beta$, i.e., the upper limit of t_i becomes $p_{i_o} + p_\alpha + p_\beta - t_\alpha - t_\beta$ and all other limits in the right member of (2.5) are unchanged.

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